Matrix Completion and Matrix Concentration

Lester Mackey[†]

Collaborators: Ameet Talwalkar[‡], Michael I. Jordan[**],
Richard Y. Chen[*], Brendan Farrell[*], and Joel A. Tropp[*]

[†]Stanford University [‡]UCLA [**]UC Berkeley
[*]California Institute of Technology

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Part I

Divide-Factor-Combine

Motivation: Large-scale Matrix Completion

Goal: Estimate a matrix $\mathbf{L}_0 \in \mathbb{R}^{m \times n}$ given a subset of its entries

$$\begin{bmatrix} ? & ? & 1 & \dots & 4 \\ 3 & ? & ? & \dots & ? \\ ? & 5 & ? & \dots & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 & 1 & \dots & 4 \\ 3 & 4 & 5 & \dots & 1 \\ 2 & 5 & 3 & \dots & 5 \end{bmatrix}$$

Examples

- Collaborative filtering: How will user i rate movie j?
  - Netflix: 10 million users, 100K DVD titles
- Ranking on the web: Is URL j relevant to user i?
  - Google News: millions of articles, millions of users
- Link prediction: Is user i friends with user j?
  - Facebook: 500 million users

Motivation: Large-scale Matrix Completion

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State of the art MC algorithms

- Strong estimation guarantees
- Plagued by expensive subroutines (e.g., truncated SVD)

This talk

- Present divide and conquer approaches for scaling up any MC algorithm while maintaining strong estimation guarantees

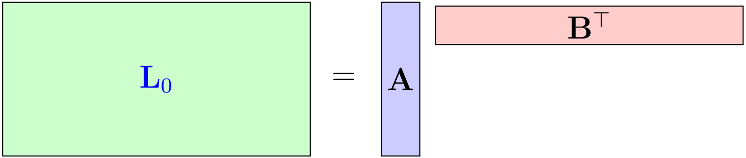
Exact Matrix Completion

Goal: Estimate a matrix $\mathbf{L}_0 \in \mathbb{R}^{m \times n}$ given a subset of its entries

Noisy Matrix Completion

Goal: Given entries from a matrix $\mathbf{M} = \mathbf{L}_0 + \mathbf{Z} \in \mathbb{R}^{m\times n}$ where $\mathbf{Z}$ is entrywise noise **and** $\mathbf{L}_0$ **has rank** $\mathbf{r} \ll m, n$, estimate $\mathbf{L}_0$

- Good news: $\mathbf{L}_0$ has $\sim (m+n)r \ll mn$ degrees of freedom



  - Factored form: $\mathbf{A}\mathbf{B}^\top$ for $\mathbf{A} \in \mathbb{R}^{m\times r}$ and $\mathbf{B} \in \mathbb{R}^{n\times r}$
- Bad news: Not all low-rank matrices can be recovered

Question: What can go wrong?

What can go wrong?

Entire column missing

$$\begin{bmatrix} 1 & 2 & ? & 3 & \dots & 4 \\ 3 & 5 & ? & 4 & \dots & 1 \\ 2 & 5 & ? & 2 & \dots & 5 \end{bmatrix}$$

- No hope of recovery!

Solution: Uniform observation model

Assume that the set of s observed entries Ω is drawn uniformly at random:

$$\Omega \sim \mathsf{Unif}(m, n, s)$$

What can go wrong?

Bad spread of information

$$\mathbf{L} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Can only recover $\mathbf{L}$ if $\mathbf{L}_{11}$ is observed

Solution: Incoherence with standard basis (Candès and Recht, 2009)

A matrix $\mathbf{L} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top \in \mathbb{R}^{m \times n}$ with $\mathrm{rank}(\mathbf{L}) = r$ is *incoherent* if

Singular vectors are not too skewed: $\begin{cases} \max_i \left\|\mathbf{U}\mathbf{U}^\top \mathbf{e}_i\right\|^2 \le \mu r/m \\ \max_i \left\|\mathbf{V}\mathbf{V}^\top \mathbf{e}_i\right\|^2 \le \mu r/n \end{cases}$

and not too cross-correlated: $\|\mathbf{U}\mathbf{V}^\top\|_\infty \le \sqrt{\frac{\mu r}{mn}}$

How do we estimate $\mathbf{L}_0$?

First attempt:

$$\begin{aligned} \text{minimize}_{\mathbf{A}} \quad & \operatorname{rank}(\mathbf{A}) \\ \text{subject to} \quad & \textstyle\sum_{(i,j)\in\Omega} (\mathbf{A}_{ij} - \mathbf{M}_{ij})^2 \leq \Delta^2. \end{aligned}$$

Problem: Computationally intractable!

Solution: Solve **convex** relaxation (Fazel, Hindi, and Boyd, 2001; Candès and Plan, 2010)

$$\begin{aligned} \text{minimize}_{\mathbf{A}} \quad & \|\mathbf{A}\|_* \\ \text{subject to} \quad & \textstyle\sum_{(i,j)\in\Omega} (\mathbf{A}_{ij} - \mathbf{M}_{ij})^2 \leq \Delta^2 \end{aligned}$$

where $\|\mathbf{A}\|_* = \sum_k \sigma_k(\mathbf{A})$ is the trace/nuclear norm of $\mathbf{A}$.

Questions:

- Will the nuclear norm heuristic successfully recover $\mathbf{L}_0$?
- Can nuclear norm minimization scale to large MC problems?

Noisy Nuclear Norm Heuristic: Does it work?

Yes, with high probability.

Typical Theorem

If $\mathbf{L}_0$ with rank r is incoherent, $s \gtrsim rn\log^2(n)$ entries of $\mathbf{M} \in \mathbb{R}^{m\times n}$ are observed uniformly at random, and $\hat{\mathbf{L}}$ solves the noisy nuclear norm heuristic, then

$$\|\hat{\mathbf{L}} - \mathbf{L}_0\|_F \leq f(m,n)\Delta$$

with high probability when $\|\mathbf{M} - \mathbf{L}_0\|_F \leq \Delta$.

- See Candès and Plan (2010); Mackey, Talwalkar, and Jordan (2011); Keshavan, Montanari, and Oh (2010); Negahban and Wainwright (2010)
- Implies **exact** recovery in the noiseless setting ($\Delta = 0$)

Noisy Nuclear Norm Heuristic: Does it scale?

Not quite...

- Standard interior point methods (Candès and Recht, 2009):
  $O(|\Omega|(m+n)^3 + |\Omega|^2(m+n)^2 + |\Omega|^3)$
- More efficient, tailored algorithms:
  - Singular Value Thresholding (SVT) (Cai, Candès, and Shen, 2010)
  - Augmented Lagrange Multiplier (ALM) (Lin, Chen, Wu, and Ma, 2009)
  - Accelerated Proximal Gradient (APG) (Toh and Yun, 2010)
  - All require rank-k truncated SVD on **every** iteration

Take away: These provably accurate MC algorithms are too expensive for large-scale or real-time matrix completion

Question: How can we scale up a given matrix completion algorithm and still retain estimation guarantees?

Divide-Factor-Combine (DFC)

Our Solution: Divide and conquer

1. Divide $\mathbf{M}$ into submatrices.
2. Factor each submatrix **in parallel**.
3. Combine submatrix estimates to estimate $\mathbf{L}_0$.

Advantages

- Factoring a submatrix is often much cheaper than factoring $\mathbf{M}$
- Multiple submatrix factorizations can be carried out in parallel
- DFC works with **any** base MC algorithm
- With the right choice of division and recombination, yields estimation guarantees comparable to those of the base algorithm

DFC-PROJ: Partition and Project

1. Randomly partition $\mathbf{M}$ into t column submatrices $\mathbf{M} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 & \cdots & \mathbf{C}_t \end{bmatrix}$ where each $\mathbf{C}_i \in \mathbb{R}^{m \times l}$
2. Complete the submatrices **in parallel** to obtain
$$\begin{bmatrix} \hat{\mathbf{C}}_1 & \hat{\mathbf{C}}_2 & \cdots & \hat{\mathbf{C}}_t \end{bmatrix}$$
   - Reduced cost: Expect t-fold speed-up per iteration
   - Parallel computation: Pay cost of one cheaper MC
3. Project submatrices onto a single low-dimensional column space
   - Estimate column space of $\mathbf{L}_0$ with column space of $\hat{\mathbf{C}}_1$
$$\hat{\mathbf{L}}^{proj} = \hat{\mathbf{C}}_1 \hat{\mathbf{C}}_1^+ \begin{bmatrix} \hat{\mathbf{C}}_1 & \hat{\mathbf{C}}_2 & \cdots & \hat{\mathbf{C}}_t \end{bmatrix}$$
   - Common technique for randomized low-rank approximation (Frieze, Kannan, and Vempala, 1998)
   - Minimal cost: $O(mk^2 + lk^2)$ where $k = \mathrm{rank}(\hat{\mathbf{L}}^{proj})$
4. **Ensemble:** Project onto column space of each $\hat{\mathbf{C}}_j$ and average

DFC: Does it work?

Yes, with high probability.

Theorem (Mackey, Talwalkar, and Jordan, 2014b)

If $\mathbf{L}_0$ with rank r is incoherent and $s = \omega(r^2 n \log^2(n)/\epsilon^2)$ entries of $\mathbf{M} \in \mathbb{R}^{m \times n}$ are observed uniformly at random, then $l = o(n)$ random columns suffice to have

$$\|\hat{\mathbf{L}}^{proj} - \mathbf{L}_0\|_F \leq (2 + \epsilon) f(m,n) \Delta$$

with high probability when $\|\mathbf{M} - \mathbf{L}_0\|_F \leq \Delta$ and the noisy nuclear norm heuristic is used as a base algorithm.

- Can sample vanishingly small fraction of columns ($l/n \to 0$)
- Implies exact recovery for noiseless ($\Delta = 0$) setting
- Analysis streamlined by matrix Bernstein inequality

DFC: Does it work?

Yes, with high probability.

Proof Ideas:

1. Uniform column/row sampling yields submatrices with low coherence (high spread of information) w.h.p.
2. Each submatrix has sufficiently many observed entries w.h.p.

$\Rightarrow$ Submatrix completion succeeds

3. Uniform sampling of columns/rows captures the full column/row space of $\mathbf{L}_0$ w.h.p.
   - Noisy analysis builds on randomized ℓ_2 regression work of Drineas, Mahoney, and Muthukrishnan (2008)

$\Rightarrow$ Column projection succeeds

DFC-Nys: Generalized Nyström Decomposition

1. Choose a random column submatrix $\mathbf{C} \in \mathbb{R}^{m\times l}$ and a random row submatrix $\mathbf{R} \in \mathbb{R}^{d\times n}$ from $\mathbf{M}$. Call their intersection $\mathbf{W}$.

$$\mathbf{M} = \begin{bmatrix} \mathbf{W} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} \mathbf{W} \\ \mathbf{M}_{21} \end{bmatrix} \quad \mathbf{R} = [\mathbf{W} \quad \mathbf{M}_{12}]$$

2. Recover the low rank components of $\mathbf{C}$ and $\mathbf{R}$ **in parallel** to obtain $\hat{\mathbf{C}}$ and $\hat{\mathbf{R}}$
3. Recover $\mathbf{L}_0$ from $\hat{\mathbf{C}}$, $\hat{\mathbf{R}}$, and their intersection $\hat{\mathbf{W}}$

$$\hat{\mathbf{L}}^{nys} = \hat{\mathbf{C}}\hat{\mathbf{W}}^{+}\hat{\mathbf{R}}$$

   - Generalized Nyström method (Goreinov, Tyrtyshnikov, and Zamarashkin, 1997)
   - Minimal cost: $O(mk^2 + lk^2 + dk^2)$ where $k = \mathrm{rank}(\hat{\mathbf{L}}^{nys})$
4. **Ensemble:** Run p times in parallel and average estimates

DFC Noisy Recovery Error

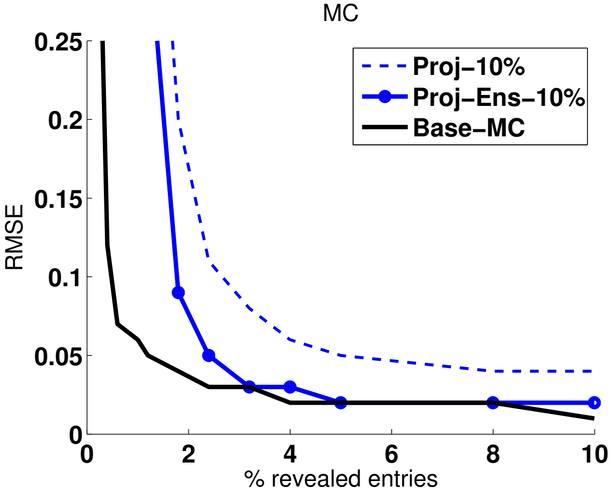


Figure : Recovery error of DFC relative to base algorithm (APG) with $m = 10K$ and $r = 10$.

DFC Speed-up

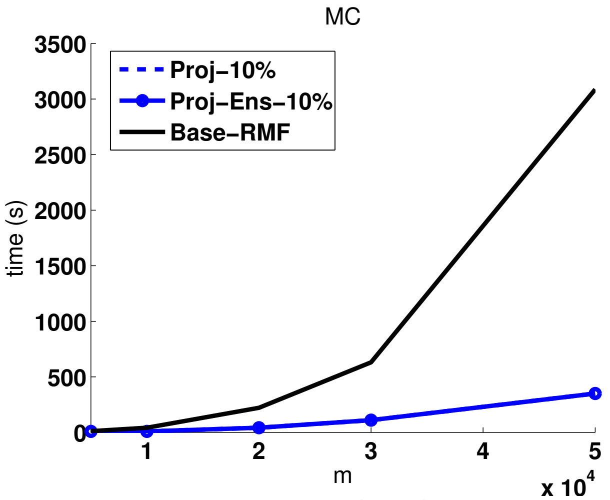


Figure : Speed-up over base algorithm (APG) for random matrices with $r = 0.001m$ and 4% of entries revealed.

Application: Collaborative filtering

Task: Given a sparsely observed matrix of user-item ratings, predict the unobserved ratings

Issues

- Full-rank rating matrix
- Noisy, non-uniform observations

The Data

- **Netflix Prize Dataset**[1]
  - 100 million ratings in $\{1, \dots, 5\}$
  - 17,770 movies, 480,189 users

[1] `http://www.netflixprize.com/`

Application: Collaborative filtering

Task: Predict unobserved user-item ratings

Method	Netflix	
	RMSE	Time
APG	0.8433	2653.1s
DFC-Proj-25%	0.8436	689.5s
DFC-Proj-10%	0.8484	289.7s
DFC-Proj-Ens-25%	0.8411	689.5s
DFC-Proj-Ens-10%	0.8433	289.7s

Beyond Matrix Completion

- Video background modeling via robust matrix factorization
  (Mackey, Talwalkar, and Jordan, 2014b)
- Image tagging / video event detection via subspace segmentation
  (Talwalkar, Mackey, Mu, Chang, and Jordan, 2013)

Part II

Stein's Method for Matrix Concentration

Concentration Inequalities

Matrix concentration

$$\mathbb{P}\{\|\boldsymbol{X} - \mathbb{E}\,\boldsymbol{X}\| \geq t\} \leq \delta$$
$$\mathbb{P}\{\lambda_{\max}(\boldsymbol{X} - \mathbb{E}\,\boldsymbol{X}) \geq t\} \leq \delta$$

- Non-asymptotic control of random matrices with complex distributions

Applications

- Matrix completion from sparse random measurements

  (Gross, 2011; Recht, 2011; Negahban and Wainwright, 2010; Mackey, Talwalkar, and Jordan, 2014b)
- Randomized matrix multiplication and factorization

  (Drineas, Mahoney, and Muthukrishnan, 2008; Hsu, Kakade, and Zhang, 2011)
- Convex relaxation of robust or chance-constrained optimization

  (Nemirovski, 2007; So, 2011; Cheung, So, and Wang, 2011)
- Random graph analysis (Christofides and Markström, 2008; Oliveira, 2009)

Concentration Inequalities

Matrix concentration

$$\mathbb{P}\{\lambda_{\max}(\boldsymbol{X} - \mathbb{E}\,\boldsymbol{X}) \geq t\} \leq \delta$$

Difficulty: Matrix multiplication is not commutative

$$\Rightarrow \mathrm{e}^{\boldsymbol{X}+\boldsymbol{Y}} \neq \mathrm{e}^{\boldsymbol{X}}\mathrm{e}^{\boldsymbol{Y}}$$

Past approaches (Ahlswede and Winter, 2002; Oliveira, 2009; Tropp, 2011)

- Rely on deep results from matrix analysis
- Apply to sums of independent matrices and matrix martingales

This work

- Stein's method of exchangeable pairs (1972), as advanced by Chatterjee (2007) for scalar concentration
  - ⇒ Improved exponential tail inequalities (Hoeffding, Bernstein)
  - ⇒ Polynomial moment inequalities (Khintchine, Rosenthal)
  - ⇒ Dependent sums and more general matrix functionals

Roadmap

Notation

Hermitian matrices: $\mathbb{H}^d = \{\boldsymbol{A} \in \mathbb{C}^{d\times d} : \boldsymbol{A} = \boldsymbol{A}^*\}$

- *All matrices in this talk are Hermitian.*

Maximum eigenvalue: $\lambda_{\max}(\cdot)$

Trace: $\operatorname{tr}\boldsymbol{B}$, the sum of the diagonal entries of $\boldsymbol{B}$

Spectral norm: $\|\boldsymbol{B}\|$, the maximum singular value of $\boldsymbol{B}$

Matrix Stein Pair

Definition (Exchangeable Pair)

(Z, Z') is an *exchangeable pair* if $(Z, Z') \stackrel{d}{=} (Z', Z)$.

Definition (Matrix Stein Pair)

Let (Z, Z') be an auxiliary exchangeable pair, and let $\boldsymbol{\Psi} : \mathcal{Z} \to \mathbb{H}^d$ be a measurable function. Define the random matrices

$$\boldsymbol{X} := \boldsymbol{\Psi}(Z) \quad \text{and} \quad \boldsymbol{X}' := \boldsymbol{\Psi}(Z').$$

$(\boldsymbol{X}, \boldsymbol{X}')$ is a *matrix Stein pair* with scale factor $\alpha \in (0, 1]$ if

$$\mathbb{E}[\boldsymbol{X}' \mid Z] = (1 - \alpha)\boldsymbol{X}.$$

- Matrix Stein pairs are exchangeable pairs
- Matrix Stein pairs always have zero mean

Method of Exchangeable Pairs

Why Matrix Stein pairs?

- Furnish more convenient expressions for moments of $\boldsymbol{X}$

Lemma

Let $(\boldsymbol{X}, \boldsymbol{X}')$ be a matrix Stein pair with scale factor α and $\boldsymbol{F} : \mathbb{H}^d \to \mathbb{H}^d$ a measurable function with $\mathbb{E}\|(\boldsymbol{X} - \boldsymbol{X}')\boldsymbol{F}(\boldsymbol{X})\| < \infty$. Then

$$\mathbb{E}[\boldsymbol{X}\ \boldsymbol{F}(\boldsymbol{X})] = \frac{1}{2\alpha}\mathbb{E}[(\boldsymbol{X} - \boldsymbol{X}')(\boldsymbol{F}(\boldsymbol{X}) - \boldsymbol{F}(\boldsymbol{X}'))]. \tag{1}$$

Intuition

- Expressions like $\mathbb{E}\left[\boldsymbol{X}\mathrm{e}^{\theta\boldsymbol{X}}\right]$ and $\mathbb{E}[\boldsymbol{X}^p]$ arise naturally in concentration settings
- Eq. 1 allows us to bound these integrals using the smoothness properties of $\boldsymbol{F}$ and the discrepancy $\boldsymbol{X} - \boldsymbol{X}'$

The Conditional Variance

Why Matrix Stein pairs?

- Give rise to a measure of spread of the distribution of $\boldsymbol{X}$

Definition (Conditional Variance)

Suppose that $(\boldsymbol{X}, \boldsymbol{X}')$ is a matrix Stein pair with scale factor α, constructed from the exchangeable pair (Z, Z'). The *conditional variance* is the random matrix

$$\boldsymbol{\Delta}_{\boldsymbol{X}} := \boldsymbol{\Delta}_{\boldsymbol{X}}(Z) := \frac{1}{2\alpha}\,\mathbb{E}\left[(\boldsymbol{X} - \boldsymbol{X}')^2 \,|\, Z\right].$$

- $\boldsymbol{\Delta}_{\boldsymbol{X}}$ is a stochastic estimate for the variance, $\mathbb{E}\,\boldsymbol{X}^2 = \frac{1}{2\alpha}\,\mathbb{E}[(\boldsymbol{X} - \boldsymbol{X}')^2] = \mathbb{E}\,\boldsymbol{\Delta}_{\boldsymbol{X}}$

Take-home Message

Control over $\boldsymbol{\Delta}_{\boldsymbol{X}}$ yields control over $\lambda_{\max}(\boldsymbol{X})$

Exponential Concentration for Random Matrices

Theorem (Mackey, Jordan, Chen, Farrell, and Tropp, 2014a)

Let $(\boldsymbol{X}, \boldsymbol{X}')$ be a matrix Stein pair with $\boldsymbol{X} \in \mathbb{H}^d$. Suppose that

$$\boldsymbol{\Delta}_{\boldsymbol{X}} \preccurlyeq c\boldsymbol{X} + v\,\mathbf{I} \quad \text{almost surely for} \quad c, v \geq 0.$$

Then, for all $t \geq 0$,

$$\mathbb{P}\{\lambda_{\max}(\boldsymbol{X}) \geq t\} \leq d \cdot \exp\left\{\frac{-t^2}{2v + 2ct}\right\}.$$

- Control over the conditional variance $\boldsymbol{\Delta}_{\boldsymbol{X}}$ yields
  - Gaussian tail for $\lambda_{\max}(\boldsymbol{X})$ for small t, exponential tail for large t
- When $d = 1$, reduces to scalar result of Chatterjee (2007)
- The dimensional factor d cannot be removed

Matrix Hoeffding Inequality

Corollary (Mackey, Jordan, Chen, Farrell, and Tropp, 2014a)

Let $\boldsymbol{X} = \sum_k \boldsymbol{Y}_k$ for independent matrices in $\mathbb{H}^d$ satisfying

$$\mathbb{E}\,\boldsymbol{Y}_k = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{Y}_k^2 \preccurlyeq \boldsymbol{A}_k^2$$

for deterministic matrices $(\boldsymbol{A}_k)_{k\geq 1}$. Define the scale parameter

$$\sigma^2 := \left\| \sum\nolimits_k \boldsymbol{A}_k^2 \right\|.$$

Then, for all $t \geq 0$,

$$\mathbb{P}\left\{ \lambda_{\max}\left(\sum\nolimits_k \boldsymbol{Y}_k \right) \geq t \right\} \leq d \cdot \mathrm{e}^{-t^2/2\sigma^2}.$$

- Improves upon the matrix Hoeffding inequality of Tropp (2011)
    - Optimal constant $1/2$ in the exponent
- Can replace scale parameter with $\sigma^2 = \frac{1}{2}\left\| \sum_k \left(\boldsymbol{A}_k^2 + \mathbb{E}\,\boldsymbol{Y}_k^2 \right) \right\|$
    - Tighter than classical Hoeffding inequality (1963) when $d = 1$

Exponential Concentration: Proof Sketch

1. **Matrix Laplace transform method** (Ahlswede & Winter, 2002)

- Relate tail probability to the *trace* of the mgf of $\boldsymbol{X}$

$$\mathbb{P}\{\lambda_{\max}(\boldsymbol{X}) \geq t\} \leq \inf_{\theta > 0} \mathrm{e}^{-\theta t} \cdot m(\theta)$$

where $m(\theta) := \mathbb{E}\,\mathrm{tr}\,\mathrm{e}^{\theta \boldsymbol{X}}$.

How to bound the trace mgf?

- Past approaches: Golden-Thompson, Lieb's concavity theorem
- Chatterjee's strategy for scalar concentration
  - Control mgf growth by bounding derivative

$$m'(\theta) = \mathbb{E}\,\mathrm{tr}\,\boldsymbol{X}\mathrm{e}^{\theta \boldsymbol{X}} \quad \text{for } \theta \in \mathbb{R}.$$

  - Perfectly suited for rewriting using exchangeable pairs!

Exponential Concentration: Proof Sketch

2. Method of Exchangeable Pairs

- Rewrite the derivative of the trace mgf

$$m'(\theta) = \mathbb{E} \operatorname{tr} \boldsymbol{X} \mathrm{e}^{\theta \boldsymbol{X}} = \frac{1}{2\alpha} \mathbb{E} \operatorname{tr} \left[(\boldsymbol{X} - \boldsymbol{X}') \left(\mathrm{e}^{\theta \boldsymbol{X}} - \mathrm{e}^{\theta \boldsymbol{X}'} \right) \right].$$

Goal: Use the smoothness of $\boldsymbol{F}(\boldsymbol{X}) = \mathrm{e}^{\theta \boldsymbol{X}}$ to bound the derivative

Mean Value Trace Inequality

Lemma (Mackey, Jordan, Chen, Farrell, and Tropp, 2014a)

Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a weakly increasing function and that $h : \mathbb{R} \to \mathbb{R}$ is a function whose derivative h' is convex. For all matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{H}^d$, it holds that

$$\operatorname{tr}[(g(\boldsymbol{A}) - g(\boldsymbol{B})) \cdot (h(\boldsymbol{A}) - h(\boldsymbol{B}))] \leq \frac{1}{2} \operatorname{tr}[(g(\boldsymbol{A}) - g(\boldsymbol{B})) \cdot (\boldsymbol{A} - \boldsymbol{B}) \cdot (h'(\boldsymbol{A}) + h'(\boldsymbol{B}))].$$

- *Standard matrix functions:* If $g : \mathbb{R} \to \mathbb{R}$ and

$$\boldsymbol{A} := \boldsymbol{Q} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{bmatrix} \boldsymbol{Q}^*, \quad \text{then} \quad g(\boldsymbol{A}) := \boldsymbol{Q} \begin{bmatrix} g(\lambda_1) & & \\ & \ddots & \\ & & g(\lambda_d) \end{bmatrix} \boldsymbol{Q}^*$$

- For exponential concentration we let $g(\boldsymbol{A}) = \boldsymbol{A}$ and $h(\boldsymbol{B}) = \mathrm{e}^{\theta \boldsymbol{B}}$
- Inequality does not hold without the trace

Exponential Concentration: Proof Sketch

3. Mean Value Trace Inequality

- Bound the derivative of the trace mgf

$$
\begin{aligned}
m'(\theta) &= \frac{1}{2\alpha}\,\mathbb{E}\,\mathrm{tr}\left[(\boldsymbol{X}-\boldsymbol{X}')\left(\mathrm{e}^{\theta\boldsymbol{X}}-\mathrm{e}^{\theta\boldsymbol{X}'}\right)\right] \\
&\le \frac{\theta}{4\alpha}\,\mathbb{E}\,\mathrm{tr}\left[(\boldsymbol{X}-\boldsymbol{X}')^2\cdot\left(\mathrm{e}^{\theta\boldsymbol{X}}+\mathrm{e}^{\theta\boldsymbol{X}'}\right)\right] \\
&= \frac{\theta}{2\alpha}\,\mathbb{E}\,\mathrm{tr}\left[(\boldsymbol{X}-\boldsymbol{X}')^2\cdot\mathrm{e}^{\theta\boldsymbol{X}}\right] \\
&= \theta\cdot\mathbb{E}\,\mathrm{tr}\left[\frac{1}{2\alpha}\,\mathbb{E}\left[(\boldsymbol{X}-\boldsymbol{X}')^2\,|\,Z\right]\cdot\mathrm{e}^{\theta\boldsymbol{X}}\right] \\
&= \theta\cdot\mathbb{E}\,\mathrm{tr}\left[\boldsymbol{\Delta}_{\boldsymbol{X}}\,\mathrm{e}^{\theta\boldsymbol{X}}\right].
\end{aligned}
$$

Exponential Concentration: Proof Sketch

3. Mean Value Trace Inequality

- Bound the derivative of the trace mgf

$$m'(\theta) \le \theta \cdot \mathbb{E}\,\mathrm{tr}\left[\boldsymbol{\Delta}_{\boldsymbol{X}}\, \mathrm{e}^{\theta \boldsymbol{X}}\right].$$

4. Conditional Variance Bound: $\boldsymbol{\Delta}_{\boldsymbol{X}} \preccurlyeq c\boldsymbol{X} + v\,\mathbf{I}$

- Yields differential inequality

$$\begin{aligned} m'(\theta) &\le c\theta\,\mathbb{E}\,\mathrm{tr}\left[\boldsymbol{X}\,\mathrm{e}^{\theta \boldsymbol{X}}\right] + v\theta\,\mathbb{E}\,\mathrm{tr}\left[\mathrm{e}^{\theta \boldsymbol{X}}\right] \\ &= c\theta \cdot m'(\theta) + v\theta \cdot m(\theta). \end{aligned}$$

- Solve to bound $m(\theta)$ and thereby bound

$$\mathbb{P}\{\lambda_{\max}(\boldsymbol{X}) \ge t\} \le \inf_{\theta > 0} \mathrm{e}^{-\theta t} \cdot m(\theta) \le d \cdot \exp\left\{\frac{-t^2}{2v + 2ct}\right\}.$$

Polynomial Moments for Random Matrices

Theorem (Mackey, Jordan, Chen, Farrell, and Tropp, 2014a)

Let $p = 1$ or $p \geq 1.5$. Suppose that $(\boldsymbol{X}, \boldsymbol{X}')$ is a matrix Stein pair where $\mathbb{E}\|\boldsymbol{X}\|_{2p}^{2p} < \infty$. Then

$$\left(\mathbb{E}\|\boldsymbol{X}\|_{2p}^{2p}\right)^{1/2p} \leq \sqrt{2p-1} \cdot \left(\mathbb{E}\|\boldsymbol{\Delta}_{\boldsymbol{X}}\|_{p}^{p}\right)^{1/2p}.$$

- **Moral:** The conditional variance controls the moments of $\boldsymbol{X}$
- Generalizes Chatterjee's version (2007) of the scalar Burkholder-Davis-Gundy inequality (Burkholder, 1973)
  - See also Pisier & Xu (1997); Junge & Xu (2003, 2008)
- Proof techniques mirror those for exponential concentration
- Also holds for infinite-dimensional Schatten-class operators

Application: Matrix Khintchine Inequality

Corollary (Mackey, Jordan, Chen, Farrell, and Tropp, 2014a)

Let $(\varepsilon_k)_{k\geq 1}$ be an independent sequence of Rademacher random variables and $(\boldsymbol{A}_k)_{k\geq 1}$ be a deterministic sequence of Hermitian matrices. Then if $p = 1$ or $p \geq 1.5$,

$$\left(\mathbb{E}\left\|\sum\nolimits_k \varepsilon_k \boldsymbol{A}_k\right\|_{2p}^{2p}\right)^{1/2p} \leq \sqrt{2p-1} \cdot \left\|\left(\sum\nolimits_k \boldsymbol{A}_k^2\right)^{1/2}\right\|_{2p}.$$

- Noncommutative Khintchine inequality (Lust-Piquard, 1986; Lust-Piquard and Pisier, 1991) is a dominant tool in applied matrix analysis
  - e.g., Used in analysis of column sampling and projection for approximate SVD (Rudelson and Vershynin, 2007)
- Stein's method offers an unusually concise proof
- The constant $\sqrt{2p-1}$ is within $\sqrt{\mathrm{e}}$ of optimal

Extensions

Refined Exponential Concentration

- Relate trace mgf of conditional variance to trace mgf of $\boldsymbol{X}$
- Yields matrix generalization of classical Bernstein inequality
- Offers tool for unbounded random matrices

General Complex Matrices

- Map any matrix $\boldsymbol{B} \in \mathbb{C}^{d_1 \times d_2}$ to a Hermitian matrix via *dilation*

$$\mathscr{D}(\boldsymbol{B}) := \begin{bmatrix} \mathbf{0} & \boldsymbol{B} \\ \boldsymbol{B}^* & \mathbf{0} \end{bmatrix} \in \mathbb{H}^{d_1+d_2}.$$

- Preserves spectral information: $\lambda_{\max}(\mathscr{D}(\boldsymbol{B})) = \|\boldsymbol{B}\|$

Dependent Sequences

- Combinatorial matrix statistics (e.g., sampling w/o replacement)
- Matrix-valued functions satisfying a self-reproducing property
  - Yields a dependent bounded differences inequality for matrices

General Exchangeable Matrix Pairs (Paulin, Mackey, and Tropp, 2014)

The End

Thanks!

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